PRICING EXOTIC OPTIONS IN THE FINITE MOMENT LOG-STABLE MODEL

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ABSTRACT. We investigate the model introduced by Carr and Wu in [13] form the perspective of exotic option pricing. The focus is set on exotic options of Lookback and Barrier type. In some cases, closed-form results are available. We also compare various simulation techniques, for the purpose of Monte-Carlo valuation.

1. Introduction

Because of its analytical tractability and its connection to the central limit theorem, the Gaussian distribution is ubiquitous in many research fields (Physics, Biology, Economy, Finance). In Finance, it was introduced by Bachelier in [3] and the subsequent developments related to option pricing and involving the Brownian motion were synthesized in [34], section 6.

However, models based on the Gaussian law are constrained by at least two of its features: its symmetry and its very light tails. Furthermore, most of the random phenomena pertaining to these models are far from normally distributed. It is therefore logical to look for alternatives. The α-stable family of distributions seems appealing since its has the Gaussian law as a limit ($\alpha \to 2$) and thus generalizes it. It has heavy tails and enables both positive and negative skewness.

Chronologically, it was probably Mandelbrot who first introduced stable distributions into Economic Theory [31] and Finance [32]. An overview of applications of stable laws in Finance is given in the monograph [39]. One of the recurrent questions which arise in the literature is whether or not stock returns have stable-like distributions. Scholars and practitioners have published contradicting results on this topic for over four decades (a non-exhaustive sample is: [16], [1], [29], [22]).

If returns can indeed be modeled by stable laws, then the next logical step is the pricing of options written on stocks driven by such distributions. In order to do so, the classical framework is to use exponentials of Lévy processes (hence postulating that returns are i.i.d., regardless of their time scale). This is quite problematic since the heavy tails of the stable laws imply infinite prices for standard Call options under these assumptions. Empirically, one way to circumvent this inconvenience is to consider options with very short maturity (see [33]). Unfortunately, this is not satisfactory since options are very often long-lived (warrants, for instance). A tractable solution was provided by the Finite Moment Log-Stable (FMLS) model, due to Carr and Lu in [13] (even though a hint towards this direction was already in [33]).

Their idea is to resort to completely asymmetric stable distributions. In this case, the left tail remains heavy, but the right tail becomes sub-exponential, thereby yielding finite option prices. Another way to ensure finite prices is to force the damping of the tails of the distribution. The result is a wide class of distribution, known as the tempered-stable (also referred to as CGMY or KoBoL) laws, which has been extensively studied in Finance ([7], [9], [11], [26] (chapter 12), [38] and [40]).

The purpose of this paper is twofold. First, we wish to further investigate the model of Carr and Wu [13] by providing exact and approximative results on barrier and lookback options. Our second aim is to make a short review of up-to-date Monte-Carlo simulation techniques for exotic options in exponential-Lévy models within the stable framework. We would like to underline that even though our results are tailored for stock markets, they can be easily be transposed to other securities (commodities, futures,

indexes or Foreign Exchange derivatives, for instance), whenever the recourse to stable distributions seems relevant.

The paper is organized as follows. In section two and three, we set the framework of the model and state our exact results. Section four is devoted to a discussion on approximative simulation methods towards valuation. One technical result, along with the proofs are given in the appendix.

2. Presentation of the model

We start by postulating that the stock value under consideration can be modeled, under the risk-neutral measure P as follows

$$(2.1) \qquad S_t = S_0 e^{(r-d-\sigma^\alpha)t + \sigma X_t^{(\alpha)}}, \quad t \geq 0$$

where r and d are the continuous risk-free rate and dividend rate respectively, which we assume to be constant; σ is a strictly positive constant and $X^{(\alpha)}$ a spectrally negative α-stable Lévy process with $\alpha \in (1, 2)$ and such that

$$(2.2) \qquad \mathbb{E}_P[e^{\sigma X_t^{(\alpha)}}] = e^{t\sigma^\alpha}$$

The model therefore has two free parameters (σ and α), while the classical Black-Scholes model has only one, the volatility. For notational convenience, we will henceforth omit the dependence in P in our notations, since all of our results will hold under the risk-neutral measure. Compared to [13], we have introduced a scaling factor which further simplifies the calculations. It is easy to see with (2.2) that $\mathbb{E}[S_t^n]$ is finite for any $n \geq 0$, which justifies the denomination of the model.

We denote by $\mathcal{F}_t$ the natural filtration of the process $X^{(\alpha)}$. Condition (2.2) ensures that the process $L_t = e^{-(r-d)t} S_t$ is an $\mathcal{F}_t$-martingale under P. An important feature of $X^{(\alpha)}$ is that it has only negative jumps, hence the distribution of the asset's log-returns $R_{T-t} = \log(S_T/S_t)$ is strongly negatively skewed: its density has a power decaying tail on the left and an exponentially decaying tail on the right. Empirically, this can be interpreted as follows: stocks usually increase slowly, with daily positive returns rarely above 5% or 10%, while they can experience massive daily losses due either to macro-economic shocks or to the publication of unfavorable stock-specific news or reports.

In their paper [13], Carr and Wu show that the representation (2.1) for the stock value is able to generate any type of configuration for the implied volatility smirks observed in the S&P 500 option market. As figures 1 and 2 show, α drives the slope and σ the level. In order to compute vanilla price options in their model, they use the method developed by Carr and Madan in [12]. With the help of this technique, they show that the post-calibration pricing error implied by (2.1) is never worse (in fact often better) than that of other popular models with 3 to 6 free parameters (for instance, the Merton Jump-Diffusion process with stochastic volatility).
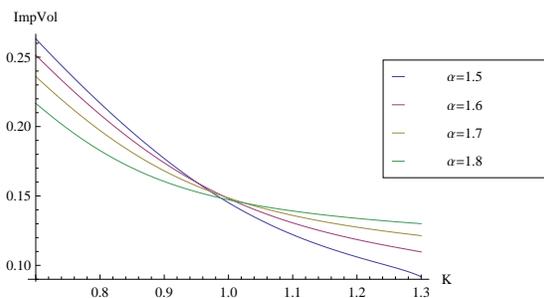
FIGURE 1. Graph of the implied volatility for $S = K = 1$, $\sigma = 0.15$ and $r - d = 0$
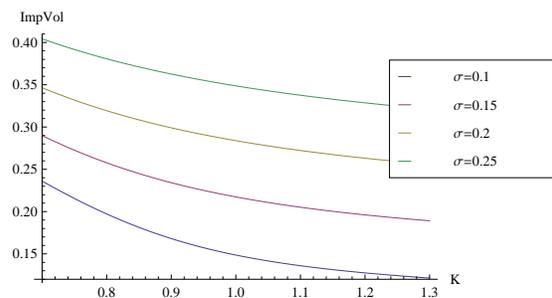
FIGURE 2. Graph of the implied volatility for $S = K = 1$, $\alpha = 1.7$ and $r - d = 0$

Nowadays, the option market for most stocks with large market capitalization is very liquid. Hence, the model calibration can be used to price options with more complicated payoffs. We will focus on two types of such products. We begin with the lookback options which have the following payoffs at maturity $t = T$

(2.3) $$p^{LC} = S_T - I_T \text{ for a Lookback Call}, \quad p^{LP} = M_T - S_T \text{ for a Lookback Put}$$

where

$$I_T = \inf_{0 \leq t \leq T} S_t, \quad M_T = \sup_{0 \leq t \leq T} S_t$$

which guarantees that the payoff is always positive. The prices at time t for these securities under the risk-neutral measure are

(2.4) $$LC_t = e^{-r(T-t)} \mathbb{E}[S_T - I_T | \mathcal{F}_t] \quad \text{and} \quad LP_t = e^{-r(T-t)} \mathbb{E}[M_T - S_T | \mathcal{F}_t]$$

The time $t = 0$ is the date of issuance of the option.

The second family of options is much wider. Barrier options are classical puts and calls which are activated or killed if a barrier has been hit (or not hit) before the maturity of the option. The option is called "In" (*resp. "Out"*) if it is activated (*resp. killed*) upon hitting the barrier. If the barrier is to be reached from below (*resp. above*), then the option is "Up" (*resp. "Down"*). For instance, if $(x)^+ = \max(x, 0)$, we define the payoff at maturity of these options in two cases

$$p^{UIP} = (K - S_T)^+ \mathbf{1}_{\{M_T > B\}} \text{ for an Up and In Put}$$

$$p^{DOC} = (S_T - K)^+ \mathbf{1}_{\{I_T > B\}} \text{ for a Down and Out Call}$$

where K is the strike of the option and B its barrier.

The corresponding prices are

(2.5) $$UIP_t = e^{-r(T-t)} \mathbb{E}[(K - S_T)^+ \mathbf{1}_{\{M_T > B\}} | \mathcal{F}_t]$$

and

$$DOC_t = e^{-r(T-t)} \mathbb{E}[(S_T - K)^+ \mathbf{1}_{\{I_T > B\}} | \mathcal{F}_t]$$

It is possible to provide exact formulae for (2.4) and (2.5) in some cases. It is the purpose of the next section.

3. Exact valuation

3.1. **Pricing lookback options - case $r - d = \sigma^\alpha$.** We first deal with the lookback options and start by a simplified case when S is modeled by the exponential of an asymmetric stable Lévy process without drift. In this case, some results are available, both for the running maximum and the running minimum of the driving Lévy process. The first valuation of Lookback options, in the Black-Scholes setting, is due to Goldman, Sosin and Gatto in [19]; another reference is [17]. In the FMLS model, the following result, the proof of which may be found in the Appendix, holds.

Proposition 3.1. *If $r - d = \sigma^\alpha$, then, for T=1,*

(3.1) $$LP_0 = S_0 e^{-r} \left(E_{1/\alpha}(\sigma) - e^{\sigma^\alpha} \right)$$

and

(3.2) $$LC_0 = S_0 e^{-r} \left(e^{\sigma^\alpha} - \frac{\alpha}{\Gamma(1/\alpha)} e^{\sigma^\alpha} \int_\sigma^\infty e^{-z^\alpha} dz \right)$$

where $E_\alpha(\cdot)$ is the Mittag-Leffler function:

(3.3) $$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(1 + \alpha k)}, \quad \alpha > 0$$

Remark 3.1. For $\alpha = 2$, $E_{1/2}(z) = e^{x^2}(\text{erfc}(-z))$ (see section 45:14 in [37]). By the Désiré-André identity, the density of the supremum M^B of the Brownian motion B_t is given by

$$f_{M_t^B} = \frac{2}{\sqrt{2\pi t}} e^{-x^2/(2t)} \mathbf{1}_{\{x \geq 0\}}, \quad t > 0$$

and the change of variable $z = \sqrt{(x - \sigma t)^2/(2t)}$ then leads to

$$\int_0^\infty \frac{2e^{\sigma x}}{\sqrt{2\pi t}} e^{-x^2/(2t)} dx = e^{\sigma^2 t/2} \int_0^\infty \frac{2}{\sqrt{2\pi t}} e^{-(x-\sigma t)^2/(2t)} = e^{\sigma^2 t/2} \text{erfc}(-\sqrt{t\sigma^2/2})$$

hence the results are consistent in the Brownian case, for $S_t = S_0 e^{\sigma\sqrt{2}B_t}$.

We provide graphs of both prices as functions of α and σ (with $S_0 = 1$, $r = \sigma^\alpha$ and $d = 0$). The values were computed using Mathematica; the series in (3.3) was truncated above $k = 400$ and the integral in (3.2) was computed using the function NIntegrate.
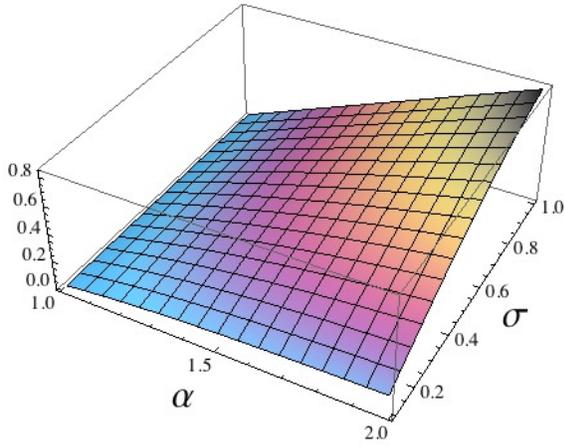
FIGURE 3. Graph of LP for $\alpha \in (1, 2)$ and $\sigma \in (0.1, 1)$
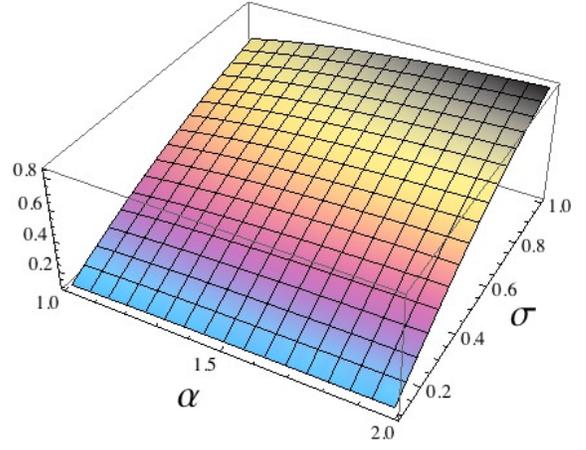
FIGURE 4. Graph of LC for $\alpha \in (1, 2)$ and $\sigma \in (0.1, 1)$

Notice that for the Lookback Put, the price is increasing, both in α and σ, but for the Lookback Call, it is almost constant in α. This difference can qualitatively be explained by the fact that a variation in α has more impact on the running minimum of S than on its running maximum.

The valuation of these products on the secondary market for $t \in (0, T)$ is a more complicated task. By the independent and stationary increments of $X^{(\alpha)}$, we can consider an updated model:

$$(3.4) \qquad\qquad S_{t+s} = S_t e^{(r-d-\sigma^\alpha)s + \sigma\tilde{X}_s^\alpha}, \quad s \geq 0$$

Where $\tilde{X}^{(\alpha)}$ is an α-stable spectrally negative Lévy process (that is, an independent copy of $X^{(\alpha)}$). The payoff of the lookback options are given by

$$p_t^{LC} = S_T - \min(I_t, I_{t,T}), \quad p_t^{LP} = \max(M_t, M_{t,T}) - S_T$$

where

$$I_{t,T} = \inf_{t \leq s \leq T} S_s, \quad M_{t,T} = \sup_{t \leq s \leq T} S_s$$

Therefore, under (3.4), the prices, at time t, of lookback options issued at time zero are detailed in the following formulae.

Theorem 3.1. *At time $t \in (0, T)$, when $r - d = \sigma^\alpha$, the lookback prices are given by,*

$$LC_t = S_t e^{(T-t)(\sigma^\alpha - r)} - e^{-r(T-t)} \sum_{n=1}^{\infty} \frac{I_t (\log(S_t/I_t)/\sigma)^{\alpha n - 1}}{\Gamma(\alpha n - 1)\Gamma(1 - n + 1/\alpha)(\alpha n - 1)(T-t)^{n-1/\alpha}}$$

$$+ S_t e^{(T-t)(\sigma^\alpha - r)} \left[\sum_{n=1}^{\infty} \frac{\int_0^{\log(S_t/I_t)/\sigma} e^{-\sigma x} x^{\alpha n - 2} dx}{\Gamma(\alpha n - 1)\Gamma(1 - n + 1/\alpha)(T-t)^{n-1/\alpha}} - \frac{\alpha}{\Gamma(1/\alpha)} \int_{\sigma(T-t)^{1/\alpha}}^{\infty} e^{-z^\alpha} dz \right]$$

$$LP_t = e^{-r(T-t)} \frac{\alpha M_t}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n/\alpha + 1)\sin(\pi n/\alpha)(\log(M_t/S_t)/\sigma)^n}{n!n(T-t)^{-n/\alpha}} - S_t e^{(T-t)(\sigma^\alpha - r)}$$

$$+ e^{-r(T-t)} S_t \left[E_{1/\alpha}[\sigma(T-t)^{1/\alpha}] - \frac{\alpha}{\pi} \sum_{n=1}^{\infty} \frac{\Gamma(n/\alpha + 1)\sin(\pi n/\alpha) \int_0^{\log(M_t/S_t)/\sigma} e^{\sigma x} x^{n-1} dx}{n!(T-t)^{-n/\alpha}} \right]$$

The proof of this result is given in the Appendix. In practice, the series must be truncated. The convergence is quite fast whenever $\log(S_t/I_t)/\sigma < 1$ or $\log(M_t/S_t)/\sigma < 1$, that is, $S_t/I_t < e^\sigma$ or $M_t/S_t < e^\sigma$. Note that the cases $S_t = I_t$ or $S_t = M_t$ are coherent with the results (3.1) and (3.2).

3.2. **Pricing of an Up and In Put - case $r = d$.** Because the computation of the expectation (2.5) requires the knowledge of the distribution of the couple (S_T, M_T), we are able to provide an exact result only in a particular case.

We follow Bowie and Carr [8]. They show that under the assumption $r = d$, barrier options can be hedged using linear combinations of vanilla options and barrier Bonds, that is, options with terminal payoffs $\mathbf{1}_{\{M_T > B\}}$, $\mathbf{1}_{\{M_T < B\}}$, $\mathbf{1}_{\{I_T > B\}}$ or $\mathbf{1}_{\{I_T < B\}}$. Unfortunately, one of their results requires a symmetry formula which is not available for stable process with $\alpha < 2$. Moreover, their second family of results depends on the Put-Call parity upon touching the barrier, therefore, the negative jumps in the FMLS model make it impossible to value "Down" type options using their methods. In fact, the only case we can consider is the Up and In Put with $K > B$. If $r = d$, it is shown in [8] that the replicating portfolio consisting of one standard call with strike K and $K - B$ Up and In Bonds is exactly equivalent to the UIP. If the barrier is never hit, both their values are zero but if the barrier is hit (continuously, i.e. without jumps), then, by the classical (vanilla) Put-Call parity, the portfolio is exactly worth the price of the vanilla put. The payoff of the digital barrier option is $\mathbf{1}_{\{M_T > B\}}$, which leads to the following valuation.

Proposition 3.2. *If $r = d$ and $K > B$, the price of an UIP option at issuance is given by*

$$\begin{aligned}
(3.5) \quad UIP_0 &= e^{-rT}(K - B)P[M_T > B] + C(K, T) \\
&= e^{-rT}(K - B)P\left[M_T^{X^{\alpha,\mu}} > \log(B/S_0)/\sigma\right] + C(K, T)
\end{aligned}$$

where $\mu = -\sigma^\alpha$ is the drift of $\log(S_T/S_0)$, $M_T^{X^{\alpha,\mu}} = \sup_{0 \leq t \leq T}(X_t^{(\alpha)} + \mu t) = \sup_{0 \leq t \leq T} \log(S_t/S_0)$. $C(K, T)$ is the price of a vanilla call of strike K and maturity T.

Of course, under the event $(M_t < B)$, the price at time $t < T$ is easily derived:

$$\begin{aligned}
(3.6) \quad UIP_t &= e^{-r(T-t)}(K - B) \int_B^{\infty} f_{\tilde{M}_{T-t}}(x)dx + C_t(K, T - t) \\
&= e^{-r(T-t)}(K - B)P\left[M_{T-t}^{\tilde{X}^{\alpha,\mu}} > \log(B/S_t)/\sigma\right] + C_t(K, T - t)
\end{aligned}$$

where $C_t(K, T - t)$ is the price, at time t, of a vanilla call with strike K and maturity $T - t$ and $M_s^{\tilde{X}^{\alpha,\mu}} = \sup_{0 \le t \le s} (\tilde{X}_t^{(\alpha)} + \mu t)$. For an absolutely continuous random variable Y, the function f_Y will denote its density.

The probabilities in (3.5) and (3.6) can be computed in a closed form using Theorem A.1 in the Appendix for a non-rational α - this is a very mild condition, since, for instance, $\alpha = 1.5 \approx \sqrt{2} + 0.086$. We underline that these results also gives the price for the corresponding UOP, by the relationship $UOP + UIP = P$. Equation (3.5) stems from an exact static hedging strategy because there are no positive jumps. It is therefore not possible to proceed similarly for the DIC options. The condition $r = d$ can be suited to options on futures or, as in [8], to foreign exchange options on the spot, when $r^d = r^f$ (both the domestic and the foreign risk free rates are equal).

4. Approximative pricing

4.1. **Classical Monte-Carlo, first method.** Whenever the jump measure of a Lévy process is known, it is possible to simulate approximative sample paths of this process and hence to generate sample payoffs of exotic options. Repeating this procedure many times and invoking the law of large numbers gives a close proxy to the average value of the payoff. This technique is usually called Monte-Carlo pricing. If the payoffs have a finite variance (which is the case for all classical exotic options in the FMLS model), the speed of convergence of this method is $\hat{\sigma}_N / \sqrt{N}$ where N is the number of simulations and $\hat{\sigma}_N$ is the empirical standard error of the N generated random payoffs. The central limit theorem provides confidence intervals for the average value of the proxy.

The critical issue in the FMLS model is that the underlying Lévy process has infinite activity: it has an infinite number of very small jumps within any time interval. In this case, the best simulation technique to date was developed by Asmussen and Rosinski in [2]. If we are aiming at simulating $X_t^{\alpha,\mu} = X_t^{(\alpha)} + \mu t$, then we should consider

$$(4.1) \qquad X_t^{\epsilon} = \mu_{\epsilon} t + v_{\epsilon} B_t + \sum_{0 \le s \le t} \Delta X_s^{(\alpha)} \mathbf{1}_{|\Delta X_s^{(\alpha)}| > \epsilon}, \quad t > 0, \, \epsilon \in (0, 1)$$

Where B_t is a standard Brownian motion. In this representation, the small jumps have been omitted and the values μ_{ϵ} and v_{ϵ} account for the mean and standard error of this truncation. In our case, because the jumps are fully compensated $\mu_{\epsilon} = \mu$, and

$$v_{\epsilon} = \left(\int_{-\epsilon}^{0} x^2 \nu(dx) \right)^{1/2} = \left(\int_{-\epsilon}^{0} \frac{x^2}{\Gamma(-\alpha)(-x)^{1+\alpha}} dx \right)^{1/2} = \frac{\epsilon^{1-\alpha/2}}{\sqrt{\Gamma(-\alpha)(2-\alpha)}}$$

In his PhD thesis, Dia, [15], proved the following error bounds, which we have adapted to our setting. The prices LC, LP and UIP are given by (2.4) and (2.5) while LC^{ϵ}, LP^{ϵ}, UIP^{ϵ} are their approximated counterparts, using the process defined by (4.1).

Proposition 4.1. *For $\epsilon \in (0, 1/2)$ and $T = 1$, the approximation errors satisfy*

$$|LP - LP^{\epsilon}| \le S_0 C \max\left(\epsilon^{2-\alpha}, \epsilon^{1-\alpha/3} \sqrt{\log(\epsilon^{-\alpha/3})} \right)$$

$$|LC - LC^{\epsilon}| \le S_0 C \max\left(\epsilon^{2-\alpha}, \epsilon^{1-3\alpha/8} \sqrt{\log(\epsilon^{-\alpha/4})} \right)$$

$$|UIP - UIP^{\epsilon}| \le S_0 C \epsilon^{1/2-\alpha/6} \sqrt{\log(\epsilon^{-\alpha/6})}$$

and the latter rate remains valid for any barrier put option.

The S_0 constant was added to remind that the error increases linearly with this variable.

Proof. In the FMLS model, the functions defined by Dia are equal to $\sigma_0(\epsilon) = \epsilon^{1-\alpha/2}$, $\beta(\epsilon) = C\epsilon^\alpha$ where C is a generic constant which does not depend on ϵ. The error on $\mathbb{E}[S_1]$ is given by Proposition 4.18 (and the remark subsequent to Proposition 4.4) and it is equal to $C\epsilon^{2-\alpha}$. It must be compared with the error on $\mathbb{E}[I_1]$ which is smaller $C\epsilon^{1-\alpha/3}\sqrt{\log(\epsilon^{-\alpha/3})}$ (Theorem 4.22 with $f(x) = e^{-x}$). Lastly, the error on $\mathbb{E}[M_1]$ is bounded by $C\epsilon^{1-3\alpha/8}\sqrt{\log(\epsilon^{-\alpha/4})}$, see Proposition 4.28 (with $p = 2$ and $\theta = 1/2$).

The result for barrier options stems from Proposition 5.50 in [15], with $\rho = \theta = 1/2$ and $\tilde{\beta}$ defined in Proposition 4.29. $\square$

These error bounds are quite problematic when α is not close to 1. For instance, if $\alpha = 1.5$, and we want to price a Lookback Put, then in order to obtain a 10^{-2} precision for $S_0 = C = 1$, we must choose $\epsilon = 2.10^{-5}$, which is very small. This corresponds to a jump intensity of 3.10^6, which means that we have to simulate, on average three million jumps per unit interval.

The exact results (3.1) and (3.2), enable us to test these bounds. We have performed Monte-Carlo simulations for various values of ϵ and α, with $\sigma = 0.5$ and $S_0 = 1$. In order to be precise, we chose $N = 10^6$ so that $\hat{\sigma}_N/\sqrt{N} \le 10^{-3}$. The errors of the prices are provided in the table below. The NA:="Not Available" cells were not computed, since, for $\alpha = 1.5$ and $\epsilon = 10^{-4}$, the simulations lasted twenty hours. The case $\alpha = 1.9$ and $\epsilon = 10^{-4}$ would have required several days, which is irrelevant in a pricing context.

	$\epsilon = 10^{-1}$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	exact value
Lookback Call ($\alpha = 1.1$)	<0.001	-0.002	-0.002	-0.002	0.419
($\alpha = 1.5$)	+0.001	-0.007	<0.001	+0.02	0.483
($\alpha = 1.9$)	+0.008	-0.002	+0.002	NA	0.515
Lookback Put ($\alpha = 1.1$)	+0.002	<0.001	+0.001	+0.001	0.066
($\alpha = 1.5$)	+0.012	+0.010	+0.001	+0.002	0.296
($\alpha = 1.9$)	+0.033	+0.023	+0.011	NA	0.481

TABLE 1. Absolute errors on lookback prices for various values of ϵ and α (Δt=0.005, $\sigma = 0.5$)

There are two main conclusions to be drawn from this table. First, the error bounds in Proposition 4.1 are not optimal. The convergence is in fact faster. The second conclusion is more technical. In order to be able to simulate the sample paths of the Brownian motion, we chose a time discretization of Δt=0.005 for the piecewise constant Euler scheme, that is to say, for $i \in [1, (\Delta t)^{-1}]$ and $t_i = i\Delta t$,

$$B_t \approx \tilde{B}_{t_i} = \sqrt{\Delta t}\sum_{k=1}^{i} N_k, \quad \forall t \in [t_i, t_{i+1})$$

where the N_k are independent normal laws. $\Delta t = 0.005$ represents 200 points per unit interval. In comparison, the case $\alpha = 1.1$ and $\epsilon = 0.1$ implies on average 1.4 jumps per unit interval while the case $\alpha = 1.5$ and $\epsilon = 0.0001$ requires on average more than 280,000 jumps per unit time. Hence, the choice of the time discretization Δt introduces a bias and should be made in accordance with ϵ. This is what may explain why, as ϵ decreases, the prices of the Lookback Call do not gain accuracy, except when $\alpha = 1.9$.

It seems appealing to think that taking $v_\epsilon = 0$ would probably have given better results in some cases. We have thus run the same computations, but without the Brownian part and the outcome is summarized in the table below.

For $\alpha = 1.1$, the errors are comparable to the simulations embedding a Brownian component. However, in the other cases, the error is quite sizable, especially for $\alpha = 1.9$. This can be explained by the fact that $v_\epsilon = 0.015$ when $\epsilon = 0.001$ and $\alpha = 1.1$, while $v_\epsilon = 0.95$ when $\alpha = 1.9$: in order to match the volatility of $X^{(\alpha)}$, the simulation requires an important Brownian coefficient when $\alpha = 1.9$. Consequently, it seems reasonable to keep the Brownian component whenever $\alpha \ge 3/2$ and $\epsilon \in (10^{-4}, 10^{-1})$.

4.2. **Classical Monte-Carlo, second method.** The second method of Monte Carlo simulation relies on the closeness of stable distribution under convolution. Indeed, if $X^{(\alpha)}$ is an $\alpha-$stable distribution and $X_1, \ldots, X_n$ n independent copies of it, then

(4.2) $$\frac{X_1 + \cdots + X_n}{n^{1/\alpha}} \stackrel{d}{=} X^{(\alpha)} + d_n$$

	$\epsilon = 10^{-1}$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$
Lookback Call ($\alpha = 1.1$)	-0.006	-0.017	-0.004	-0.001
($\alpha = 1.5$)	+0.028	+0.003	+0.005	+0.015
($\alpha = 1.9$)	+0.287	+0.216	+0.173	NA
Lookback Put ($\alpha = 1.1$)	+0.007	+0.006	+0.001	<0.001
($\alpha = 1.5$)	+0.075	+0.029	+0.005	+0.003
($\alpha = 1.9$)	+0.321	+0.238	+0.179	NA

TABLE 2. Absolute error on lookback prices for various values of ϵ and α ($\sigma = 0.5$, no Brownian component)

for some real number d_n.

If we consider the process $X^{(\alpha)}$, then $\forall t \geq 0$, $\mathbb{E}[X_t^{(\alpha)}] = 0$ and $d_n = 0$ (the random variable is in fact strictly stable, see [41], Definition 13.1 and Theorem 14.7 (vi). Because of the independence of its increments, is therefore possible to simulate $X^{\alpha,\mu}$ using a discrete uniform skeleton $t_i = i\Delta t = i/n$ for $i = 0, \ldots n$ with $t_0 = 0$ and $t_n = T$. The approximation $\hat{X}^{\alpha,\mu}$ is a piecewise constant process such that $\tilde{X}_0^{\alpha,\mu} = Y_0 = 0$ and

$$(4.3) \qquad \begin{cases} Y_i &= Y_{i-1} + Z_i^{(\alpha)}/n^{1/\alpha}, \quad i \in [1,n] \\ \tilde{X}_t^{\alpha,\mu} &= Y_i + i\mu T/n, \quad \forall t \in [t_i, t_{i+1}), \ i \in [0,n] \end{cases}$$

where the $Z_i^{(\alpha)}$ are independent and have a completely asymmetric $\alpha-$distribution (i.e., they have the same law as $X_1^{(\alpha)}$).

We use the method of Chambers, Mallows and Stuck [14] to simulate the stable random variables, that is

$$(4.4) \qquad Z^{(\alpha)} = \frac{\sin\left(\alpha\left(U + \frac{\pi(2-\alpha)}{2\alpha}\right)\right)}{\cos(U)^{1/\alpha}} \left(\frac{\cos\left(U - \alpha\left(U + \frac{\pi(2-\alpha)}{2\alpha}\right)\right)}{E}\right)^{(1-\alpha)/\alpha}$$

is $\alpha-$asymmetrically distributed if U is a uniform random variable on $[-\pi/2, \pi/2]$ and E is an independent $1-$exponentially distributed variable.

This simulation method has two advantages over the previous one: first, there is no interference between the simulation of the pure-jump part and that of the Brownian part; second, the number of simulation points is deterministic.

We provide below some results on the valuation of two barrier options. The first one is an Up and In Put with $S_0 = 40$, $B = 45$, $K = 50$ and $T = 1$, while the second one is a Down and Out Call with $S_0 = 50$ and $K = 40$. In order to compare our results to those of subsection 3.2, we consider the case $r = d$. The parameters are $\sigma = 1/2$, $\alpha = 3/2$, and we set $r = d = 0$. We first put the stress on the effect of n on the convergence of the price of the barrier from a discretely monitored process to a quasi-continuously observed process. The number of simulations is N=200,000 in all of the cases.

Let us focus on the UIP. Because the simulation is stepwise constant, the actual behavior of $X^{\alpha,\mu}$ inside the interval (t_i, t_{i+1}) is unknown and the supremum of the process inside this time interval is very likely to be greater than $X_{t_i}^{\alpha,\mu}$ or $X_{t_{i+1}}^{\alpha,\mu}$. This is why, once $i = n$ and the discretization is over, both S_T and $-I_T$ are underestimated by such a procedure. As n increases, the range of this underestimation decreases and the event $\{\tilde{S}_T > B\}$ (embedded in the payoff) becomes slightly more likely. This explains why the price of the UIP increases as n increases. The opposite effect is of course logical in the case of the DOC.

Up and In Put ($\alpha = 1.5$)	$n = 50$	$n = 150$	$n = 500$	$n = 2000$
Average	8.389	8.851	9.145	9.298
Std Error ($\hat{\sigma}_N$)	12.795	13.047	13.223	13.154
CPU time (sec.)	5	14	44	174
Down and Out Call ($\alpha = 1.5$)	$n = 50$	$n = 150$	$n = 500$	$n = 2000$
Average	10.909	10.535	10.345	9.986
Std Error ($\hat{\sigma}_N$)	21.953	21.799	21.864	21.453
CPU time (sec.)	5	14	44	174

TABLE 3. MC results on barrier option prices for various values of n

We wish to underline that formula (3.5) combined to Theorem A.1 yields a value of 9.38 for the UIP (with $\alpha = \sqrt{2} + 0.086 \approx 3/2$). This is 0.08 above our mean value for $n = 2000$ (we recall that $\hat{\sigma}_N/\sqrt{N} \approx 0.03$), hence the convergence to a continuous observation is quite slow.

For $\alpha = \sqrt{15/4} \approx 1.93$, the approximative value for the UIP is 11.286 (with the same parameters and $n = 2000$), for a theoretical value of 11.892. For $\alpha = \sqrt{5/4} \approx 1.118$, the values are 3.304 versus 3.283. The convergence (as n increases) is thus faster as $\alpha \downarrow 1$.

4.3. **Wiener-Hopf Monte-Carlo.** A recent method, described in [25], also makes it possible to avoid the technical arbitrage between Δt and ϵ of the first method. The idea is to concatenate small samples of independent trajectories of $X^{\alpha,\mu}$ stopped at an independent exponential time and to use the Wiener-Hopf factorization (see [6], chapter VI or [27], chapter 6). More precisely, if we consider I_{e_q} and S_{e_q} the running infimum and supremum of $X^{\alpha,\mu}$ stopped at time e_q, an independent $q-$exponentially distributed time, then $X^{\alpha,\mu}_{e_q} \stackrel{d}{=} I^{X^{\alpha,\mu}}_{e_q} + M^{X^{\alpha,\mu}}_{e_q}$. If we define $t(q,n) = \sum_{k=1}^{n} e_q^{(i)}$ where the $e_q^{(i)}$ are independent copies of e_q, then it is formally proved in [25] that

$$(4.5) \qquad \left(X^{\alpha,\mu}_{t(q,n)}, S^{\alpha,\mu}_{t(q,n)} \right) \stackrel{d}{=} (V(q,n), J(q,n))$$

where V and J are defined iteratively for $n \geq 1$ by

$$V(q,n) \;=\; V(q, n-1) + I^{(n)}_{e_q} + M^{(n)}_{e_q}$$
$$J(q,n) \;=\; \max\left(J(q, n-1), V(q, n-1) + M^{(n)}_{e_q} \right)$$

where $V(q,0) = J(q,0) = 0$ and $M^{(n)}_{e_q}$ (resp. $I^{(n)}_{e_q}$) are independent copies of $M^{X^{\alpha,\mu}}_{e_q}$ (resp. $I^{X^{\alpha,\mu}}_{e_q}$). This construction is in fact fairly natural as it corresponds to a simple concatenation of independent trajectories of X.

Invoking the law of large numbers, we then have, for k large enough

$$\mathbb{E}\left[F\left(X^{\alpha,\mu}_t, M^{X^{\alpha,\mu}}_t \right) \right] \approx \frac{1}{k} \sum_{m=1}^{k} F(V^{(m)}(q,n), J^{(m)}(q,n))$$

where $V^{(m)}(q,n)$ and $J^{(m)}(q,n)$ are independent copies of $V(q,n)$ and $J(q,n)$ under the obvious condition $\mathbb{E}[t(q,n)] = n/q = t$.

Theoretically, this technique seems very appealing. In practice, however, things are more complicated. This algorithm is only efficient when it is possible to simulate $I^{(n)}_{e_q}$ and $M^{(n)}_{e_q}$ very quickly, which is not the case for stable processes. $M^{(n)}_{e_q}$ is of course not the problem, since, by Corollary 2, Chapter VII in [6], it is an exponential variable with a parameter which is easy to compute. There is, nonetheless no simple way to simulate $I^{(n)}_{e_q}$. One can proceed

    $\circ$ either with the acceptance-rejection method; but it requires a companion distribution with a density that behaves like that of $I^{(n)}_{e_q}$. This is problematic since not only does $f_{I^{X^{\alpha,\mu}}_t}$ go to

infinity when $x \downarrow 0$ - this could have been handled with a gamma distribution -, but the tail of $I_t^{X^{\alpha,\mu}}$ is also of polynomial type (see (A.2) in the Appendix and (2.62 in [5]). There is, to our knowledge, no easy-to-simulate random variable with such characteristics.

○ or with the c.d.f. inversion technique, once e_q has been drawn. In this case, a truncation of the series is required and even with an enhanced Newton-Raphson algorithm, this method is quite lengthy (at least one hundred loops to compute $P[I_t^{X^{\alpha,\mu}} \leq x]$ for a single value x...).

○ in both cases, only the density and c.d.f. of $I_t^{X^{\alpha,\mu}}$ is known and only in the driftless case ($\mu = 0$). The random variable e_q must be drawn first. The major issue is that since we can only work with truncated series, the computation of (A.2) may explode if just one sample value of $t \stackrel{d}{=} e_q$ is too small.

Even though the Wiener-Hopf Monte Carlo method can prove to be very efficient when the stopped supremum and infimum are easily simulated, it seems that it is in fact less tractable than classical Monte-Carlo techniques in the case of stable processes.

4.4. **A word on Quasi Monte-Carlo.** In order to increase the speed of convergence of the simulation methods described above, a popular solution is Quasi-Monte Carlo (QMC) and the use of sequences with low discrepancy. We refer to chapter 5 in [18] for technical details.

The pricing of exotic options using QMC was investigated in [28] (section 5) and [21]. Before we discuss our numerical results, we wish to comment on these references. Once the parameters of the process are fixed, two choices remain: the number of simulations N and the number of points in the simulation grid (n in (4.3), τ in [28] and d in [21]). The maximum value for N is 80,000 in [28] and around 120,000 in [21]. An intuitive property, which is observed in both articles (see tables 2 in [28] and 5.3 in [21]), is that the competitiveness of QMC methods decreases as n increases.

In practice, exotic options are discretely monitored. The monitoring can be monthly, weekly, daily, etc. Therefore, the less frequent the monitoring is, the more relevant the QMC methods become. We underline that QMC methods require *a priori* the knowledge of the number of random variables to be simulated, this is why it is not suited to techniques relying on jump diffusions or on the rejection method.

Using (4.3) and (4.4), have computed the price of an Up and In Put with $S_0 = 40$, $B = 45$ and $K = 50$ and of a Down and Out Call with $S_0 = 50$, $B = 45$ and $K = 40$. In both cases, we fixed $\alpha : 3/2$ and $\sigma = 1/2$ in order to compare with the results of the classical Monte-Carlo procedure in Table 3. The pseudo random numbers were generated by a $2n$ dimension Sobol sequence: the first n numbers being used for the uniform variable and the last n numbers for the exponential variable.

We compare MC and QMC methods in the graphs below.

For vanilla payoffs, it is well known (see [18], chapter 5) that the convergence of QMC methods is $O(\log(N)^n/N)$ while it is $O(N^{-1/2})$ for MC methods. Figures 5 and 6 in [28] illustrate this feature. However, for path-dependent payoffs, the competitiveness of QMC versus MC is (much) less obvious. As a rule of thumb, it seems that the prices are close to stable for $N > 50,000$ in the Sobol case, and when $N > 100,000$ for the classical Monte-Carlo method. QMC thus appears slightly more effective than MC, but requires a few extra seconds of computation.

4.5. **PIDE methods.**   Another family of methods for computing barrier option prices in exponential Lévy models consists in solving Partial Integro-Differential Equations (PIDE). A short review of these techniques is given in the introduction of [23]. When the small jumps are replaced by the Brownian component, Kudryavstev and Levendorskii show that the error can be quite sizable. In the case of infinite activity, they strongly recommend not to truncate the small jumps but rather to resort to the Wiener-Hopf factorization of the underlying Lévy process.

However, these methods do not apply for stable processes, for two reasons:

○ the Wiener-Hopf factorization for stable processes is very complicated in the general case (see [24] for instance) and only available in the driftless case. In the spectrally negative case, the
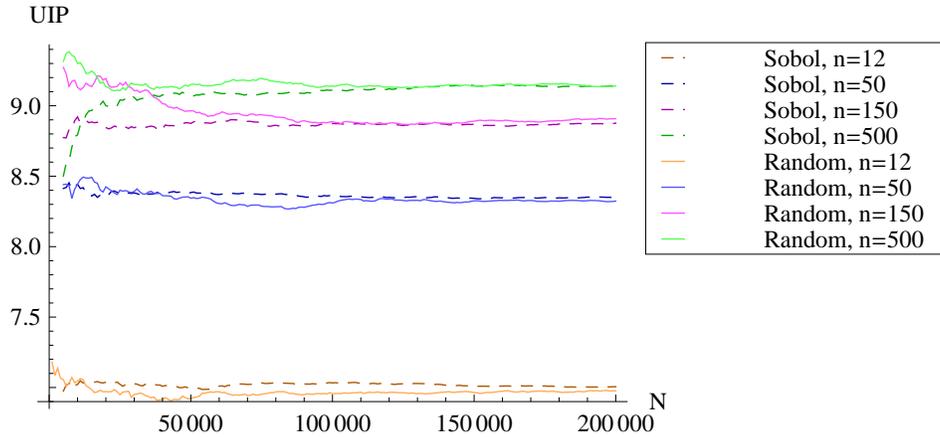
FIGURE 5. Graph of the UIP price for $n \in \{12, 50, 150, 500\}$ and $N \in (5\,000, 200\,000)$
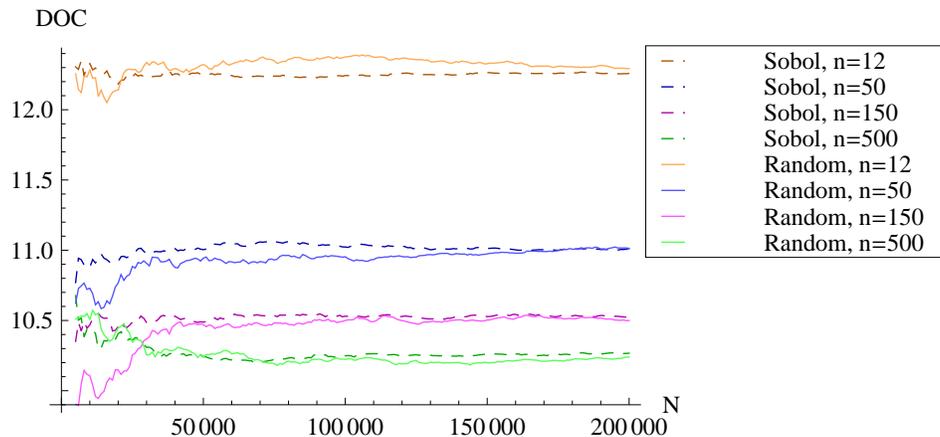
FIGURE 6. Graph of the DOC price for $n \in \{12, 50, 150, 500\}$ and $N \in (5\,000, 200\,000)$

Wiener-Hopf factors are given by Theorem 4, section VII in [6], but the inverse Lévy-Khintchine exponent Φ is only known in a completely closed-form for a very limited number of α.
- stable processes do not belong to the class of "regular Lévy processes of exponential type" (see section 2.3 in [23]) and are therefore not suited to these techniques.

5. Conclusion

This article can be used as a toolbox for anyone willing to use the FMLS as a basis for exotic option pricing. Some exact formulae are provided, but most of the time, approximations should be used for exotic option pricing. In this case, we strongly recommend our second Monte-Carlo method if no exact formula exists.

Appendix A. Technical results and proofs

We recall that X^α is a Lévy process with absolutely continuous jump measure

$$(A.1) \qquad\qquad \nu(dx) = \frac{\mathbf{1}_{\{x<0\}}}{\Gamma(-\alpha)(-x)^{1+\alpha}} dx$$

Its Lévy-Khintchine representation is indeed given by

$$\log(\mathbb{E}[e^{\sigma X_1}]) = \int_{-\infty}^0 (e^{\sigma x} - 1 - \sigma x)\nu(dx) = \sigma^\alpha$$

Because its jumps are fully compensated, X^α is a martingale. The prices (2.4) require the knowledge of the law of the supremum and infimum of X_t^α, or more generally of $X_t^{\alpha,\mu} = X_t^\alpha + \mu t$. Some results in this direction are given in [36], but they are not exactly what we seek here. We further recall the positivity parameter of X^α:

$$\rho = P[X_t^\alpha \geq 0] = (1 + (2-\alpha)/\alpha)/2 = \frac{1}{\alpha}$$

Lastly, we recall some notations: I_t, M_t are the infimum and supremum of the price process S_t while I_t^X and M_t^X are the infimum and supremum of the underlying Lévy process. X can be X^α, $X^{\alpha,\mu}$, $\tilde{X}^\alpha$ or $\tilde{X}^{\alpha,\mu}$. The two ladder processes are independent copies of X^α, $X^{\alpha,\mu}$ which are used for the pricing on the secondary market.

In our proofs, we will use the series representation of stable densities extensively. Our main source is [41], proofs can be found in [42].

Proof of Proposition 3.1. Because $r - d = \sigma^\alpha$, we only need to consider X^α (without any drift). Note that by the self-similary property of the process, it is sufficient to work with $T = 1$. Moreover, the same property yields

$$P[M_1^{X^\alpha} \leq x] = P[T_x \geq 1] = P[(T_1)^{-1/\alpha} \leq x]$$

hence $M_1^{X^\alpha}$ and $(T_1)^{-1/\alpha}$ have the same distribution, where $T_a = \inf\{t > 0, X_t^\alpha > a\}$ is the first passage time of X^α over a fixed level $a > 0$.

Because X^α has no positive jumps, then we can apply Theorem 1, section VII from [6] to get that T_a is a subordinator with characteristic exponent $\log(\mathbb{E}[e^{-zT_a}]) = -az^{1/\alpha}$, thus an $1/\alpha$-subordinator. Equation (3.1) then stems from exercise 29.18 in [41] (see also exercise 6.6 in [27]).

Equation (3.2) is a much deeper result, which is the combination of (2.55) in [5] and the fact that for any process X, $\inf_{0 \leq t \leq T} X_t = - \sup_{0 \leq t \leq T} - X_t$. □

Proof of Theorem 3.1. We start by recalling the densities of $-I^{X^\alpha}$ and M^{X^α}: for $t > 0$, $x > 0$,

$$(A.2) \qquad\qquad \frac{P\left[-I_t^{X^\alpha} \in dx\right]}{dx} = \sum_{n=1}^\infty \frac{1}{\Gamma(\alpha n - 1)\Gamma(1 - n + 1/\alpha)} \frac{x^{\alpha n - 2}}{t^{n-1/\alpha}}$$

$$(A.3) \qquad\qquad \frac{P\left[M_t^{X^\alpha} \in dx\right]}{dx} = \frac{\alpha}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \frac{\Gamma(n/\alpha + 1)}{n!} \sin(\pi n/\alpha) x^{n-1} t^{n/\alpha}$$

The first identity is simply (2.54) in [5]. For $t = 1$, the second identity stems from the first term of (3.1) and equation (2.10.9) from [42]. The self-similarity property gives the formula for any $t > 0$.

Then, for the Lookback Call, when $(s > t)$,

$$\mathbb{E}\left[\min\left(I_t, e^{\sigma I_s^{\tilde{X}^\alpha}}\right)\right] = \int_0^\infty \min(I_t, S_t e^{-\sigma x}) f_{-I_s^{\tilde{X}^\alpha}}(x) dx$$

$$= I_t \int_0^{-\log(I_t/S_t)/\sigma} f_{-I_s^{\tilde{X}^\alpha}}(x) dx + S_t \int_{-\log(I_t/S_t)/\sigma}^\infty e^{-\sigma x} f_{-I_s^{\tilde{X}^\alpha}}(x) dx$$

the first term is computed by integrating term by term (A.2). This is possible if we integrate the first term of (A.2) separately and then consider a normally converging series. For the second term, we use the decomposition

$$\int_{-\log(I_t/S_t)/\sigma}^{\infty} e^{-\sigma x} f_{-I_{\hat{s}}^{\tilde{X}^\alpha}}(x) dx = \int_0^{\infty} e^{-\sigma x} f_{-I_{\hat{s}}^{\tilde{X}^\alpha}}(x) dx - \int_0^{-\log(I_t/S_t)/\sigma} e^{-\sigma x} f_{-I_{\hat{s}}^{\tilde{X}^\alpha}}(x) dx$$

and, as in Proposition 3.1,

$$\int_0^{\infty} e^{-\sigma x} f_{-I_{\hat{s}}^{\tilde{X}^\alpha}}(x) dx = \frac{\alpha e^{s\sigma^\alpha}}{\Gamma(1/\alpha)} \int_{\sigma s^{1/\alpha}}^{\infty} e^{-z^\alpha} dz$$

the second integral can be expressed in terms of the upper incomplete gamma function, but we have chosen to leave it unchanged in the formula.

The formula for the Lookback Put can be obtained using the exact same steps and the density (A.3). □

We now provide a formula for the computation of the cumulative distribution function of the running supremum of $X^{\alpha,\mu}$.

Theorem A.1. *If α is not a rational number, then for $x, t > 0$ and $\mu \neq 0$,*

$$P\left[M_t^{X^{\alpha,\mu}} \leq x\right] = \frac{\alpha}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n/\alpha+1)}{n!\, n} \sin(\pi n/\alpha) \, _2F_1\left(1-n, -\frac{n}{\alpha}; 1 - \frac{n}{\alpha}; \frac{\mu t}{x}\right) t^{-n/\alpha} x^n$$

where $_2F_1$ is the hypergeometric function (see chapter 60 in [37]).

Before proving the theorem, we underline the fact that the function $x \mapsto {}_2F_1(1-n, y; z; x)$ is polynomial with degree $n-1$.

Proof. We keep the same notations as above, with X^α replaced by $X_t^{\alpha,\mu} = X_t^\alpha + \mu t$. Because $P[M_t^{X^{\alpha,\mu}} \leq x] = P[T_x \geq t]$, we will work with the first passage time T_x of $X^{\alpha,\mu}$ over the level $x > 0$. Since $X^{\alpha,\mu}$ has no positive jumps, we can use Corollary 3, section VII from [6]:

$$\frac{P[T_x \in dt]}{dt} = \frac{x}{t} \frac{P[X_t^{\alpha,\mu} \in dx]}{dx}$$

and the series representation of the density of $X^{\alpha,\mu}$ which is given by 14.28 and 14.30 in [41], yielding

$$\frac{P[T_x \in dt]}{dt} = \frac{x}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n/\alpha+1)}{n!} \sin(\pi n/\alpha) t^{-n/\alpha-1} (x - \mu t)^{n-1}$$

The integration of this function on a finite interval should be handled carefully. The classical argument, which invokes the normal convergence of the series, does not hold here. Instead, we consider the partial sum

$$S_k(t) = \sum_{n=1}^{k} (-1)^{n-1} \frac{x}{\pi} \frac{\Gamma(n/\alpha+1)}{n!} \sin(\pi n/\alpha) t^{-n/\alpha-1} (x - \mu t)^{n-1}, \quad t > 0$$

which makes sense since the Stirling formula implies that the term $\Gamma(n/\alpha+1)/\Gamma(n+1)$ mitigates any power term c^n at infinity (the term of the series is in fact $o(n^{-\gamma})$ for any $\gamma > 1$, as $n \to \infty$). For $0 < a < b < \infty$, it is plain that not only does $S_k(t) \to S_\infty(t) = P[T_x \in dt]/dt$ for any $t \in [a, b]$ but also that $|S_k(t)|$ is bounded for any $k \geq 1$ and $t \in [a, b]$. It is thus possible to apply the Arzela-Osgood theorem (see [30] and the references therein) in order to integrate $S_k(t)$ term by term and let $k \to \infty$.

It can then be shown (using 3.194 in [20], or the properties from subsection 2.1.2 in [4]) that the application

$$F_n := F_{n,\alpha,x,\mu} : t \mapsto \alpha x^{n-1} t^{-n/\alpha} \, _2F_1\left(1-n, -\frac{n}{\alpha}; 1 - \frac{n}{\alpha}; \frac{\mu t}{x}\right), \quad t > 0$$

is one anti-derivative of $t \mapsto t^{-n/\alpha-1}(x - \mu t)^{n-1}$, which yields

$$(A.4) \qquad P[T_x \in [t, u]] \quad = \quad \frac{x}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n/\alpha + 1)}{n!} \sin(\pi n/\alpha)(F_n(t) - F_n(u))$$

$$(A.5) \qquad\qquad\qquad\quad = \quad G_x(t) - G_x(u)$$

with

$$G_x : u \mapsto \frac{x}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n/\alpha + 1)}{n!} \sin(\pi n/\alpha) F_n(u), \quad x, u > 0$$

where the series is absolutely convergent for any $x, u > 0$ because of the asymptotics of the Hypergeometric function (see equation (9) from subsection 2.3.2 in [4]). Equation (A.5) implies that $G_x(\cdot)$ is both bounded (in $[0, 1]$) and decreasing. We thus have

$$\lim_{u \to \infty} G_x(u) = P[T_x = +\infty] \in [0, 1)$$

which is strictly positive if $\mu < 0$ ($X_t \to -\infty$ a.s. when $t \to \infty$) and equal to zero if $\mu \geq 0$. This can easily be shown when $\mu > 0$ using Wald's identity on X_{T_x} (implying $\mathbb{E}[T_x] < \infty$ in this case). If $\mu = 0$, then X oscillates (see Th.12, section VI in [6]) and thus touches x at some point in time. Therefore, $P[T_x \geq t] = G_x(t)$. $\qquad\square$

Lastly, for the sake of completeness, we wish to point out that a result exists for the supremum of a drifted spectrally positive stable process (see [35]). It can be used to compute $P[I_t^{\alpha,\mu} \leq x]$ for $x < 0$. However, because of the negative jumps, $X_{\hat{T}_{-x}}^{\alpha,\mu} \neq -x$ ($\hat{T}_{-x} = \inf\{t > 0, X_t \leq -x\}$) and the above reasoning does not apply for Down and In Calls. Nevertheless, these formulae can be used to (numerically) compute the prices of the lookback options in the general case.

If we denote by $\epsilon(N)$ the absolute value of the error induced by a truncation of the series above $N-1$. We recall that a quadratic irrational number is an irrational number that is solution to a quadratic equation with integer coefficients.

Theorem A.2. *If α is a quadratic irrational number and $\mu t \neq x$, then*

$$\epsilon(N) \leq$$

Proof. First, since $|(-1)^{n+1} \sin(\pi n/\alpha)| \leq 1$,

$$(A.6) \qquad \epsilon(N) \leq \frac{\alpha}{\pi} \sum_{n=N}^{\infty} \frac{\Gamma(n/\alpha + 1)}{n!n} \left| {}_2F_1\left(1 - n, -\frac{n}{\alpha}; 1 - \frac{n}{\alpha}; \frac{\mu t}{x}\right) \right| t^{-n/\alpha} x^n.$$

Then, using the identity $x\Gamma(x) = \Gamma(x + 1)$,

$$\left| {}_2F_1\left(1 - n, -\frac{n}{\alpha}; 1 - \frac{n}{\alpha}; \frac{\mu t}{x}\right) \right| = \left| \sum_{k=0}^{n-1} (1 - n)_k \frac{\Gamma(k - n/\alpha)}{\Gamma(-n/\alpha)} \frac{\Gamma(1 - n/\alpha)}{\Gamma(k + 1 - n/\alpha)} \frac{(\mu t)^k}{k! x^k} \right|$$

$$= \left| \sum_{k=0}^{n-1} (1 - n)_k \frac{-n/\alpha}{k - n/\alpha} \frac{(\mu t)^k}{k! x^k} \right|$$

$$(A.7) \qquad\qquad\qquad \leq \sum_{k=0}^{n-1} \frac{|(1 - n)_k|}{k!} \frac{n}{|n - \alpha k|} \frac{(\mu t)^k}{x^k}.$$

We must now resort to a classical result in Diophantine Approximation, which is given in [10] (Th. 1.2) and states that since α is an irrational quadratic number, there exists a constant $c(\alpha)$ such that

$$|\alpha - n/k| > \frac{c(\alpha)}{k^2}, \quad \forall k, n \in \mathbb{N}^*.$$

Combining this inequality with (A.6) and (A.7), we get

$$
(A.8) \qquad \epsilon(N) \leq \frac{\alpha}{\pi c(\alpha)} \sum_{n=N}^{\infty} \frac{\Gamma(n/\alpha+1)}{n!} t^{-n/\alpha} x^n \left(1 + \sum_{k=1}^{n-1} \frac{|(1-n)_k|}{(k-1)!} \frac{(\mu t)^k}{x^k} \right)
$$

Furthermore,

$$
\begin{aligned}
\frac{|(1-n)_k|}{(k-1)!} &= \frac{(n-1) \times (n-2) \times \cdots \times (n-k)}{1 \times 2 \times \cdots \times (k-1)} \\
&= \frac{(n-1)!}{(k-1)!(n-k-1)!} \\
(A.9) \qquad &= \frac{\Gamma(n)}{\Gamma(k)\Gamma(n-k)},
\end{aligned}
$$

which reaches its maximum at $k = n/2$, and for $\mu t/x \neq 1$, (A.8) reduces to

$$
(A.10) \qquad \epsilon(N) \leq \frac{\alpha}{\pi c(\alpha)} \sum_{n=N}^{\infty} \frac{\Gamma(n/\alpha+1)}{\Gamma(n/2)^2} t^{-n/\alpha} x^n \left(\frac{(\mu t/x)^n - 1}{\mu t/x - 1} \right).
$$

Now, as $z \to +\infty$, $\Gamma(z) \sim \sqrt{2\pi/e}(z/e)^{z-1/2}$ (see section 1.18 in [4]), which implies

$$
\frac{\Gamma(n/\alpha+1)}{\Gamma(n/2)^2} = \frac{n}{\alpha} \frac{\Gamma(n/\alpha)}{\Gamma(n/2)^2} \sim \frac{\sqrt{e}}{\alpha\sqrt{2\pi}} \frac{(2e)^n}{(\alpha e)^{n/\alpha}} n^{n/\alpha+1-n}, \quad n \to \infty.
$$

Since for $\mu t/x \neq 1$,

$$
\frac{(2e)^n}{(\alpha e)^{n/\alpha}} t^{-n/\alpha} x^n \left(\frac{(\mu t/x)^n - 1}{\mu t/x - 1} \right) n^{(n/\alpha+1-n)/2} \to 0, \quad n \to \infty,
$$

there is N^* such that

$$
(A.11) \qquad \frac{1}{\pi c(\alpha)} \frac{\sqrt{e}}{\sqrt{2\pi}} \frac{(2e)^n}{(\alpha e)^{n/\alpha}} t^{-n/\alpha} x^n \left(\frac{(\mu t/x)^n - 1}{\mu t/x - 1} \right) n^{(n/\alpha+1-n)/2} \leq 1, \quad \forall n \geq N^*
$$

and hence

$$
\begin{aligned}
\forall N \geq N^*, \quad \epsilon(N) &\leq \sum_{n=N}^{\infty} n^{(n/\alpha+1-n)/2} \\
&\leq \sum_{n=N}^{\infty} N^{(n/\alpha+1-n)/2} \\
&\leq \frac{N^{1+\frac{N}{2}(-1+1/\alpha)}}{\sqrt{N} - \sqrt{N}^{1/\alpha}}
\end{aligned}
$$

If $\mu t/x = 1$, then, in (A.10) and (A.11), $\frac{(\mu t/x)^n - 1}{\mu t/x - 1}$ is simply replaced replaced by n.

$\square$

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