Risk aggregation for the generalized logistic distribution

Guillaume Coqueret[a]

[a] *ESSEC Business School, Avenue Bernard Hirsch 95000 Cergy-Pontoise France*

Abstract

We analyze the tail of the sum of two random variables with exponential tails on $\mathbb{R}_+$ (exponential distribution) and $\mathbb{R}$ (generalized logistic distribution). The tails can be characterized when the dependence structure are copulas which satisfy some technical conditions. Consequences on the Value-at-Risk are derived and examples are discussed.

Keywords: Risk aggregation, Value-at-Risk, Generalized logistic distribution, Tail behavior

1. Introduction

Risk aggregation is now a classical topic for researchers and practitioners both in Finance and Insurance. The regulatory frameworks of Basel (I, II and III) and Solvency (I and II) make it crucial to understand how the various risks within a portfolio combine in order to estimate future losses. The naive way to proceed is to add up individual risks. This is of course not the best solution since assets are usually correlated, especially in the midst of financial crises.

A very popular approach seeks to analyze the marginal behavior of the risks separately from their dependence structure. This is rendered possible via a powerful tool: copulas. If $F_1, \dots, F_n$ are the cumulative distribution functions (c.d.f.) of the real-valued random variables (r.v.) $X_1, \dots, X_n$, then the copula related to these r.v.s is the unique mapping C such that

$$P[X_1 \leq x_1, \dots, X_n \leq x_n] = C(F_1(x_1), \dots, F_n(x_n)), \quad (x_1, \dots, x_n) \in \mathbb{R}^n.$$

We refer to the monograph [22] for more details on the subject. Once the marginals and the copula are specified, it becomes possible, though often complicated, to study the behavior of $P[X_1 + \cdots + X_n > x]$, as $x \to \infty$. It seems intuitive that this behavior should match, in some sense, that of the random variable which has the heaviest tail. In fact, this is very often true, especially when heavy tails are involved.

Email address: guillaume.coqueret@essec.edu (Guillaume Coqueret)

More precisely, when X_1 and X_2 have the same heavy-tailed distribution, then except for a handful of cases (see for instance Theorem 2.10 in [2]),

$$\lim_{x\to\infty} \frac{P[X_1+X_2>x]}{P[X_1>x]} = c > 0. \tag{1}$$

When X_1 and X_2 have non identical heavy tailed distribution, then this results usually holds if X_1 follows the law with the heaviest tail.

We have carried out a non-exhaustive survey of results in this direction which we have compiled in Table 1 (where MDA stands for Maximum Domain of Attraction). The references are ordered based on the assumptions mentioned in the articles. Some of the results hold for two random variables, others for $n \geq 2$, some for distributions on $\mathbb{R}$, others on $\mathbb{R}_+$. A very large majority of the findings deal with heavy-tailed distribution.

Topic	**References**
Regularly varying marginals	[2], [3], [5], [9], [8], [18], [19], [27]
Subexponential marginals	[12], [13], [17]
Lognormal marginals	[2], [4]
Marginals which belong to a MDA	[3], [18], [21], [27]
Rapidly varying marginals	[20], [21]
Regularly varying Archimedean copula	[3], [5], [9], [27]
Conditional independence	[12], [15], [20]
Asymptotic independence	[8], [13], [17], [19], [21]

Table 1: Articles for which (1) holds

The remainder of the paper is structured as follows. In Section 2, etc.

2. Sums of independent variables

We will henceforth consider two types of random variables with exponential decay. We will denote exponential distributions by the letter X, that is $X_i^{\lambda_i} \stackrel{d}{=} \mathcal{E}(\lambda_i)$ will have density $f_i(x) = \lambda_i e^{-\lambda_i x} 1_{\{x\geq 0\}}$ and c.d.f. $F_i(x) = 1 - e^{-\lambda_i x}$.

Similarly, $Y_j^{\alpha_j} \stackrel{d}{=} GL(\alpha_j)$ will denote a generalized logistic distribution with density and c.d.f. given by

$$g_j(x) = \alpha_j \frac{e^{-x}}{(1+e^{-x})^{1+\alpha_j}}, \quad G_j(x) = (1+e^{-x})^{-\alpha_j}.$$

It is therefore obvious that $Y_1^{\alpha_1}$ has the following tails

$$G_1(x) \sim e^{\alpha_1 x}, \quad x \downarrow -\infty, \tag{2}$$

$$1 - G_1(x) \sim \alpha_1 e^{-x}, \quad x \uparrow \infty, \tag{3}$$

where the notation $f(x) \sim g(x)$ stands for $\lim f(x)/g(x) \to 1$. We will also use $f(x) \approx g(x)$ when there are $c, C > 0$ such that $cg(x) \le f(x) \le Cg(x)$, asymptotically.

Further, we have

$$\mathbb{E}[e^{zY_1^\alpha}] = \frac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha)}, \quad \mathbb{E}[Y_1^\alpha] = \psi(\alpha) - \psi(1),$$

and

$$\mathbb{E}[(Y_1^\alpha)^2] - \mathbb{E}[Y_1^\alpha]^2 = \psi'(\alpha) + \psi'(1),$$

where ψ is the digamma function: $\psi(x) = \frac{d}{dx}\log\Gamma(x)$.

Given this asymmetry in the left and right tails in (2) and (3) , it will be interesting to study the aggregation problem not only for $Y_i^{\alpha_i}$ but for $-Y_i^{\alpha_i}$ as well. The rate of decrease of the tail of both the exponential and the generalized logistic distributions can be intercalated between that of the Gaussian law and that of sub-exponential distributions. Depending on the time lag at which financial returns are considered, their tail behaviors may vary (see [25]). For daily to monthly returns, the distribution of log-returns is likely to have tails with exponential decay. The exponential and generalized logistic distributions are therefore suitable tools for the study of risk aggregation in financial markets for instance. The exponential distribution may serve as a model for the loss distribution of an asset, while the generalized logistic law can closely match the distribution of financial returns.

Note that a scaled shift $x \mapsto (x-\mu)/\sigma$ in the c.d.f. gives the law of $\sigma X + \mu$ and hence the model becomes more flexible effortlessly. For notational convenience, we shall however henceforth set $\mu = 0$ and $\sigma = 1$.

Before stating one of our main results, we prove the following lemma which will be of use throughout the paper.

Lemma 2.1. *The hypergeometric function has the following asymptotic behaviors*

- *If $b > a > 0$ and $b - a \notin \mathbb{N}$, then, as $z \downarrow -\infty$,*

$$\begin{aligned}{}_2F_1(a,b;c;z) = &\frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}\left((-z)^{-a} + \frac{a(a+1-c)}{a+1-b}\frac{(-z)^{-a}}{z}\right) \\ &+ \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}\left((-z)^{-b} + \frac{b(b+1-c)}{b+1-a}\frac{(-z)^{-b}}{z}\right) + O(|z|^{-2-a}).\end{aligned}$$

- *If $b \ge a > 0$ and $b = a + m$ where $m \in \mathbb{N}$, then as $z \downarrow -\infty$,*

$$\begin{aligned}{}_2F_1(a, a+m, c, z) = &(-z)^{-a}\frac{\Gamma(c)}{\Gamma(a+m)}\sum_{n=0}^{m-1}\frac{(a)_n\Gamma(m-n)}{\Gamma(c-a-n)n!}z^{-n} \\ &+ \frac{\Gamma(c)(a)_m(a+1-c)_m}{\Gamma(c-a)\Gamma(a+m)m!}(-z)^{-a-m}(\log(-z) + h(a,c,m)) \\ &+ O((-z)^{-a-m-1}\log(-z)).\end{aligned}$$

where $(a)_n$ *is the Pochhammer symbol:* $(a)_n = \Gamma(a+n)/\Gamma(a)$, $\sum_{n=0}^{-1} = 0$ *and*

$$h(a, c, m) = \psi(1+m) + \psi(1) - \psi(a+m) - \psi(c-a-m).$$

Proof. The first point results from the combination of equation (17) p.63 in [6] and the fact that ${}_2F_1(a, b, c, z) = 1 + \frac{ab}{c}z + o(z)$ as $|z| \downarrow 0$ whenever $c \neq 0$. The second point stems directly from equation (18) p.63 in [6] . □

The convolution of logistic distributions was studied in [14] when $\alpha = 1$ and in [23] in the general case, but we present below our own representation of its c.df. in the proof of the following theorem.

Theorem 2.1. *If all the following random variables are mutually independent,*

i) $P[X_1^\lambda + X_2^\lambda > z] = \lambda z e^{-\lambda z}[1 + (\lambda z)^{-1} + O(z^{-2})]$, $z \to \infty$,

ii) $P[X_1^{\lambda_1} + X_2^{\lambda_2} > z] = \dfrac{\lambda_1}{\lambda_1 - \lambda_2} e^{-\lambda_2 z} + \dfrac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 z}$, $\forall z \in \mathbb{R}$, $0 < \lambda_1 < \lambda_2$,

iii) $P[Y_1^{\alpha_1} + Y_2^{\alpha_2} > z] \sim \alpha_1 \alpha_2 z e^{-z}$, $z \to \infty$, $\forall \alpha_1, \alpha_2 > 0$,

iv) $P[-Y_1^\alpha - Y_2^\alpha > z] = \alpha e^{-\alpha z}(z + h(1+\alpha, 1+2\alpha, 0)) + O(z e^{-(1+\alpha)z})$, $z \to \infty$

v) if $0 < \alpha_1 < \alpha_2$, $b - a \notin \mathbb{N}^*$, *then as* $z \to \infty$,

$$\begin{aligned} P[-Y_1^{\alpha_1} - Y_2^{\alpha_2} > z] =& \frac{\Gamma(\alpha_2 - \alpha_1)\Gamma(1+\alpha_1)}{\Gamma(\alpha_2)} e^{-\alpha_1 z} + \frac{\Gamma(\alpha_1 - \alpha_2)\Gamma(1+\alpha_2)}{\Gamma(\alpha_1)} e^{-\alpha_2 z} \\ &+ O(e^{-(1+\alpha_1)z}). \end{aligned}$$

Proof. (i) It is well known that if $X_1^\lambda \overset{d}{=} X_2^\lambda = \mathcal{E}(\lambda)$, then $X_1^\lambda + X_2^\lambda$ has density $f_{\Gamma(2,\lambda)}(x) = \lambda^2 x e^{-\lambda x}$, $x \geq 0$ and c.d.f. $F_{\Gamma(2,\lambda)}(x) = \gamma(2, \lambda x)/\Gamma(2)$, where γ is the lower incomplete gamma function. Given Eq. 9.2(6) in [7] and the fact that $\Gamma(2) = 1$, this yields (i).

(ii) However, if $X_i^\lambda \overset{d}{=} \mathcal{E}(\lambda_i)$ and $\lambda_1 < \lambda_2$, then

$$\begin{aligned} P[X_1^{\lambda_1} + X_2^{\lambda_2} \leq x] &= \int_{\mathbb{R}} \lambda_1 e^{-\lambda_1 x} 1_{\{x \geq 0\}} (1 - e^{-\lambda_2(z-x)}) 1_{\{x \leq z\}} dx \\ &= 1 - e^{-\lambda_1 z} - \lambda_1 \int_0^z e^{-(\lambda_1 - \lambda_2)x} dx \\ &= 1 + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 z} + \frac{\lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_1 z} \end{aligned}$$

(iii)-(v) We first compute the c.d.f. of $Y_1^{\alpha_1} + Y_2^{\alpha_2}$:

$$\begin{aligned} P[Y_1^{\alpha_1} + Y_2^{\alpha_2} \leq z] &= \int_{\mathbb{R}} f_1(x) F_2(z-x) dx \\ &= \alpha_1 \int_{\mathbb{R}} \frac{e^{-x}}{(1+e^{-x})^{\alpha_1+1} (1+e^{-(z-x)})^{\alpha_2}} dx \\ &= \alpha_1 \int_{\mathbb{R}} \frac{e^{-x(1+\alpha_2)}}{(1+e^{-x})^{\alpha_1+1} (e^{-x}+e^{-z})^{\alpha_2}} dx \\ &= \alpha_1 e^{-z} B(1+\alpha_2, \alpha_1)\, {}_2F_1 \left(1+\alpha_1, 1+\alpha_2; 1+\alpha_1+\alpha_2; 1-e^{-z}\right) \\ &= \frac{\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)}{\Gamma(1+\alpha_1+\alpha_2)} e^{-z}\, {}_2F_1 \left(1+\alpha_1, 1+\alpha_2; 1+\alpha_1+\alpha_2; 1-e^{-z}\right) \end{aligned} \tag{4}$$

where $B(\cdot,\cdot)$ is the usual beta function and ${}_2F_1$ is the hypergeometric function. The second-to-last equality stems from formula (22) p.121 in [11]. The asymptotic behaviors in (iv) and (v) are a simple application of the second (for $m=0$) and first points (respectively) of Lemma 2.1.

In order to prove (iii), we combine Eq. (22) in [6] with (4) to obtain, for $z>0$,

$$P[Y_1^{\alpha_1}+Y_2^{\alpha_2} \leq z] = \frac{\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)}{\Gamma(1+\alpha_1+\alpha_2)} e^{\alpha_1 z}\, {}_2F_1 \left(\alpha_1, 1+\alpha_1; 1+\alpha_1+\alpha_2; 1-e^{z}\right)$$

□

3. Risk aggregation under the Bernstein copula

$$\begin{aligned} F^*(z) = P[X_1+X_2 \leq z] &= \int_{\mathbb{R}} P[X_1 \in dx, X_2 \leq z-x] \\ &= \int_{\mathbb{R}} f_1(x) C_1(F_1(x), F_2(z-x)) dx, \quad z \in \mathbb{R} \end{aligned} \tag{5}$$

where f_1 is the density of X_1 and $C(\cdot,\cdot)$ is a bidimensional copula with

$$C_1(u,v) := \frac{\partial}{\partial u} C(u,v), \quad (u,v) \in [0,1]^2. \tag{6}$$

The function F^* therefore measures the probability that the return of the portfolio is less than z. If we fix a confidence interval of $p \in (0,1)$, then there is a $1-p$ probability that the return of X_1+X_2 will be less than the Value-at-Risk (VaR) defined by

$$\mathrm{VaR}_p^{X_1+X_2} = F^{*-1}(1-p), \quad p \in (0,1).$$

The additivity of the VaR has thoroughly been studied in [9] when the losses have a heavy tail. In such a framework, there is a threshold for p beyond which either

$$\mathrm{VaR}_p^{X_1+X_2} < \mathrm{VaR}_p^{X_1} + \mathrm{VaR}_p^{X_2} \quad \text{or} \quad \mathrm{VaR}_p^{X_1+X_2} > \mathrm{VaR}_p^{X_1} + \mathrm{VaR}_p^{X_2},$$

and, in any case,

$$\lim_{p\to 1} \frac{\mathrm{VaR}_p^{X_1+X_2}}{\mathrm{VaR}_p^{X_1} + \mathrm{VaR}_p^{X_2}} = c \in (0,\infty).$$

The main objective of this paper is to compare the left tail of F^* and F when X_1 and X_2 follow the same generalized logistic distribution. In some cases, we will also be able to compute the limiting value of the ratio $\mathrm{VaR}_p^{X_1+X_2}/(\mathrm{VaR}_p^{X_1} + \mathrm{VaR}_p^{X_2})$ when $p \to 1$.

3.1. Main results

We start with the tails behavior of the sum of two independent generalized logistic distributions.

Theorem 3.1. *F^* in (4) satisfies*

- $F^*(z) \sim \dfrac{\Gamma(\alpha_2-\alpha_1)\Gamma(1+\alpha_1)}{\Gamma(\alpha_2)} e^{\alpha_1 z}, \quad z \downarrow -\infty,$ *if* $\alpha_1 < \alpha_2$
- $F^*(z) \sim -z\alpha_1 e^{\alpha_1 z}, \quad z \downarrow -\infty,$ *if* $\alpha_1 = \alpha_2$

We will now consider a specific configuration for the underlying copula, depending on one assumption on C_1 defined in (6). More preciely, we are interested in the case where

$$(\mathbf{H1}) \quad \text{there are } c_{i,j} \geq 0 \quad \text{such that} \quad C_1(u,v) = \sum_{1\leq i,j\leq k} c_{i,j} v^i u^{j-1}, \quad c_{1,1} > 0,$$

Our main result characterizes the behavior of $P[X_1 + X_2 \leq z]$ under (**H1**). It can be stated as follows.

Theorem 3.2. *Under* (**H1**), *if* $-X_1 \stackrel{d}{=} \mathcal{L}(\alpha_1)$ *and* $-X_2 \stackrel{d}{=} \mathcal{L}(\alpha_2)$ *with* $\alpha_1 \leq \alpha_2$,

- *if* $\alpha_1 < \alpha_2$,

$$P[X_1 + X_2 \leq z] \sim \kappa^* P[X_1 \leq z], \quad z \downarrow -\infty$$

- *if* $\alpha_1 = \alpha_2$,,

$$P[X_1 + X_2 \leq z] \sim \kappa^{**} z P[X_1 \leq z], \quad z \downarrow -\infty$$

where $\kappa^* = \displaystyle\sum_{i=1}^{k} c_{i,1} B(\alpha_1, 1+\alpha_2 i) \frac{\Gamma(1+\alpha_1+\alpha_2 i)\Gamma(\alpha_2 i - \alpha_1)}{\Gamma(1+\alpha_2 i)\Gamma(\alpha_2 i)}$ *and*

$$\kappa^{**} = -c_{1,1} B(1+\alpha_1, \alpha_1) \frac{\Gamma(1+2\alpha_1)}{\Gamma(\alpha_1)\Gamma(1+\alpha_1)}$$

Proof. Under condition (**H1**),

$$
\begin{aligned}
F^*(z) &= \int_{\mathbb{R}} f_1(x) \sum_{1\leq i,j\leq k} c_{i,j} F_1(x)^{j-1} F_2(z-x)^i dx \\
&= \sum_{1\leq i,j\leq k} c_{i,j} \int_{\mathbb{R}} f_1(x) F_1(x)^{j-1} F_2(z-x)^i dx \\
&= \sum_{1\leq i,j\leq k} c_{i,j} \int_{\mathbb{R}} g_1(-x)(1-G_1(-x))^{j-1}(1-G_2(x-z))^i dx \\
&= \sum_{1\leq i,j\leq k} c_{i,j} \int_{\mathbb{R}} g_1(-x) \sum_{k=0}^{j-1} \binom{j-1}{k} (-G_1(-x))^k \sum_{l=0}^{i} \binom{i}{l} (-G_2(x-z))^l dx
\end{aligned}
$$

Where we have used the binomial formula for the last equality. This yields

$$
F^*(z) = \alpha_1 \sum_{1\leq i,j\leq k} c_{i,j} \left[\sum_{k=0}^{j-1} \sum_{l=0}^{i} \binom{j-1}{k} \binom{i}{l} (-1)^{k+l} \delta_{k,l}(z) \right] \tag{7}
$$

where

$$
\begin{aligned}
\delta_{k,l}(z) &= \int_{\mathbb{R}} \frac{e^x}{(1+e^x)^{\alpha_1+1}(1+e^x)^{\alpha_1 k}(1+e^{z-x})^{\alpha_2 l}} dx \\
&= \int_{\mathbb{R}} \frac{e^x}{(1+e^x)^{\alpha_1(1+k)+1}(1+e^{z-x})^{\alpha_2 l}} dx \\
&= \int_{\mathbb{R}} \frac{e^{x(1+\alpha_2 l)}}{(1+e^x)^{\alpha_1(1+k)+1}(e^x+e^z)^{\alpha_2 l}} dx
\end{aligned}
$$

and

$$
\delta_{k,l}(z) = e^z B(1+\alpha_2 l, \alpha_1(k+1))\, {}_2F_1 \left(\begin{array}{c} 1+\alpha_1(k+1), \\ 1+\alpha_2 l; \\ 1+\alpha_1(k+1)+\alpha_2 l; \\ 1-e^z \end{array} \right),
$$

which can be proven in the same fashion as Proposition **??**. Property (22) p.64 in [6] allows further to write

$$
\delta_{k,l}(z) = e^{-\alpha_1 z(k+1)} B(1+\alpha_2 l, \alpha_1(k+1))\, {}_2F_1 \left(\begin{array}{c} 1+\alpha_1(k+1), \\ \alpha_1(k+1); \\ 1+\alpha_1(k+1)+\alpha_2 l; \\ (e^z-1)/e^z \end{array} \right),
$$

Now, the asymptotic behavior of the hypergeometric function can be obtained via the transformation formulas of equations (17) and (18) p.63 in [6]. The outcome is the following:

- if $a < b$, $a < c$,

$$ {}_2F_1\left(a, b; c; 1 - e^{-z}\right) \sim \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} e^{az}, \quad z \downarrow -\infty. $$

- if $a = b < c$,

$$ {}_2F_1\left(a, a; c; 1 - e^{-z}\right) \sim -\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} z e^{az}, \quad z \downarrow -\infty. $$

Since $c_{1,1} > 0$, then if $\alpha_1 = \alpha_2$, the first term in (**??**) behaves like $-Cze^{\alpha_1 z}$, while all of the others decrease at least at the speed of $e^{\alpha_1 z}$. If $\alpha_1 \neq \alpha_2$, then the slowest possible pace of decrease is $e^{\alpha_1 z}$ and it is matched for all the terms such that $j = 1$.

The proof of the result when (**H2**) holds follows the same reasoning. □

We now turn to the analysis that this result has on the Value-at-Risk. We will stick to the case when (**H1**) holds. There are two possible situations:

- **case 1**: X_1 has heavier tails and

$$ P[X_1 + X_2 \leq z] \sim cP[X_1 \leq z] \sim ce^{\alpha_1 z}, \quad z \downarrow -\infty $$

- **case 2**: $X_1 \overset{d}{=} X_2$ and

$$ P[X_1 + X_2 \leq z] \sim -czP[X_1 \leq z] \sim -cze^{\alpha_1 z}, \quad z \downarrow -\infty $$

Proposition 3.1. *In **case 1**,*

$$ \lim_{p \to 1} \frac{\mathrm{VaR}_p^{X_1+X_2}}{\mathrm{VaR}_p^{X_1} + \mathrm{VaR}_p^{X_2}} = \frac{\alpha_2}{\alpha_1 + \alpha_2} \in (1/2, 1), \tag{8} $$

*and in **case 2**,*

$$ \lim_{p \to 1} \frac{\mathrm{VaR}_p^{X_1+X_2}}{\mathrm{VaR}_p^{X_1} + \mathrm{VaR}_p^{X_2}} = \frac{1}{2} \tag{9} $$

In all situations, the model is therefore asymptotically subadditive, in the sense of (2.1) in [10].

Before we start the proof of the Proposition, we introduce W_{-1}, the Lambert W function which is defined only for negative arguments. This function is one of the the reciprocals of the mapping $y \mapsto ye^y$. It is strictly decreasing on $(-1/e, 0)$. We refer to [26] for more details on the subject.

Proof. We only provide the details for **case 2**, as it is the most difficult one. First note that if

$$ G(z) = P[X_1 + X_2 \leq z] \sim -cze^{\alpha_1 z}, \quad z \downarrow -\infty, $$

then

$$G^{-1}(z) \sim W_{-1}(-\alpha_1 z/c)/\alpha_1, \quad z \to 0. \tag{10}$$

Indeed, if we set $G^{-1}(z) = W_{-1}(-\alpha_1 z/c)(1+h(z))$, then when $z \to 0$, $G^{-1}(z) \downarrow -\infty$ and we thus have

$$\begin{aligned} 1 = \frac{G \circ G^{-1}(z)}{z} &\sim -\frac{c}{z} G^{-1}(z) e^{\alpha_1 G^{-1}(z)}, \quad z \to 0 \\ &= -\frac{c}{\alpha_1 z}\left((1+h(z))W_{-1}(-\alpha_1 z/c)e^{(1+h(z))W_{-1}(-\alpha_1 z/c)}\right) \\ &= -\frac{c(1+h(z))}{\alpha_1} e^{h(z)W_{-1}(-\alpha_1 z/c)} \times \left(\frac{-\alpha_1}{c}\right), \end{aligned} \tag{11}$$

which goes to 1 if and only if $h(z) \to 0$, hence (10) is proved. Moreover, by Equation (17) in [26], $W_{-1}(z) \sim \log(-z)$, for $z \uparrow 0^-$. Using the same reasoning, we get that if

$$F(z) \sim e^{\alpha_1 z}, \quad z \downarrow -\infty,$$

then

$$F^{-1}(z) \sim \log(z)/\alpha_1, \quad z \to 0.$$

Since $\log(x)/\log(cx) \sim 1$ for any $c > 0$ when $x \to 0$,

$$\frac{G^{-1}(z)}{2F^{-1}(z)} \sim \frac{1}{2}, \quad z \to 0,$$

and the result is proved. In **case 1**, it is easy to show that

$$G^{-1}(z) \sim \log(z)/\alpha_1, \quad F_1^{-1}(z) + F_2^{-1}(z) \sim \frac{\alpha_1 + \alpha_2}{\alpha_2 \alpha_2} \log(z), \quad z \to 0,$$

where F_1 and F_2 are the c.d.f.s of X_1 and X_2, respectively. □

3.2. An example

In this section, we present classical families copulas for which either (**H1**) or (**H2**) holds. We of course recall that the independent copula verifies both of these assumptions.

We start with the Farlie-Gumbel-Morgestern copula, which is defined by

$$C_t^{FGM}(u,v) = uv + tuv(1-u)(1-v), \quad t \in [-1,1]$$

with

$$\frac{\partial}{\partial u} C_t^{FGM}(u,v) = (1+t-2tu)v + (2tu-t)v^2.$$

Obviously, this copula satisfies (**H1**) with

$$\begin{aligned} c_{1,1} &= 1+t \\ c_{1,2} &= -2t \\ c_{2,1} &= -t \\ c_{2,2} &= 2t \end{aligned}$$

and in this case, the behavior of G for large negative values can be precisely computed via Theorem 3.2 for any α_1, α_2, or t.

In order to numerically illustrate the result of Proposition 3.1, we have computed the ratio $\frac{\mathrm{VaR}_p^{X_1+X_2}}{\mathrm{VaR}_p^{X_1}+\mathrm{VaR}_p^{X_2}}$ for various values of $\alpha = \alpha_1 = \alpha_2$, t and p. The results are presented in Table 2.

		$t = -0.5$	$t = 0.2$	$t = 0.8$
$\alpha = 1/5$	$1 - p = 10^{-2}$	0.684	0.731	0.762
	$1 - p = 10^{-4}$	0.613	0.645	0.664
	$1 - p = 10^{-6}$	0.585	0.609	0.623
$\alpha = 1$	$1 - p = 10^{-2}$	0.634	0.698	0.739
	$1 - p = 10^{-4}$	0.597	0.637	0.658
	$1 - p = 10^{-6}$	0.577	0.606	0.620
$\alpha = 5$	$1 - p = 10^{-2}$	-0.802	-0.436	-0.233
	$1 - p = 10^{-4}$	0.347	0.417	0.456
	$1 - p = 10^{-6}$	0.468	0.510	0.530

Table 2: Ratios of VaR for various values of α, t and p

First of all, the negative numbers stem from the fact that for $p = 0.01$ and $\alpha = 5$, the tail is so light that the VaR of the portfolio is positive instead of being negative, as we expect it to be. This can be observed on Figure **??**: as α increases, the first centile shifts to the right; it seems that this shift is accelerated for the law of $X_1 + X_2$.

Next, the convergence to $1/2$ is in fact quite slow. In practice, $1-p$ is seldom below 0.01 or 0.001. Further computations show that for $\alpha_1 = \alpha_2 \in (0, 2)$, the VaR ratio will lie between $1/2$ and 1, for $p > 0.9$.

Before we turn to copulas satisfying a looser condition, we wish to highlight that Theorem 3.2.10 in [22] provides a generalization of the FGM copula which also verifies (**H1**). Another example of copulas satisfying (**H1**) is the Bernstein family (see [24] for instance).

4. Bounds for the left tail

$$\textbf{(H2)} \quad \begin{cases} \text{there are } c_i^+ \geq c_i^- \geq 0, \text{ with } c_1^- > 0 \text{ such that} \\ \forall (u,v) \in [0,1]^2, \quad \sum_{i=1}^n c_i^- v^i \leq C_1(u,v) \leq \sum_{i=1}^n c_i^+ v^i. \end{cases}$$

A natural corollary of our main result is the following

Proposition 4.1. *If $\alpha_1 < \alpha_2$, then under (H2),*

$$P[X_1 + X_2 \leq z] \approx P[X_1 \leq z], \quad z \downarrow -\infty$$

and if $\alpha_1 < \alpha_2$,

$$P[X_1 + X_2 \leq z] \approx -zP[X_1 \leq z], \quad z \downarrow -\infty,$$

The Ali-Mikhail-Haq copula is defined by

$$C_t^{AMH}(u,v) = \frac{uv}{1 - t(1-u)(1-v)}, \quad t \in [-1,1),$$

which yields

$$\frac{\partial}{\partial u} C_t^{AMH}(u,v) = \frac{v(1-t) + v^2 t}{(1 - t(1-u)(1-v))^2}.$$

In order to ensure the positivity of the coefficients $c_2^{\pm}$, we will restrict ourselves to $t \in (0,1)$. In this case, it is easy to show that

$$\begin{aligned} c_1^- &= 1 - t \\ c_1^+ &= (1-t)^{-1} \\ c_2^- &= t \\ c_2^+ &= t(1-t)^{-2}. \end{aligned}$$

When $t \in (0,1)$, it is obvious that C_t^{AHM} satisfies (**H2**).

A last example is copula #10 in the list of Archimedean copulas in [22]. It is given by

$$C_t^{\#10}(u,v) = \frac{uv}{(1 + (1-u^t)(1-v^t))^{1/t}}, \quad t \in (0,1]$$

and hence

$$\begin{aligned} \frac{\partial}{\partial u} C_t^{\#10}(u,v) &= v \left\{ (2 - v^t)(1 + (1-u^t)(1-v^t))^{-1-1/t} \right\} \\ &= v\, l_t(u,v) \end{aligned}$$

with

$$\frac{\partial}{\partial u} l_t(u,v) = (t+1)u^{t-1}(1-v^t)(2-v^t)(1 + (1-u^t)(1-v^t))^{-2-1/t} \geq 0$$

and

$$\frac{\partial}{\partial u} l_t(u,v) = v^{t-1}(1 + (1-u^t)(1-v^t))^{-2-1/t}(2 - v^t - 2u^t - tu^t - (uv)^t).$$

From the first derivative, we get that l_t reaches its maximum and minimum on the border of the unit square. From the second derivative, we get that $v \mapsto l_t(0,v)$ is increasing while $v \mapsto l_t(1,v)$ is decreasing. Therefore, the extrema can only be located at points $(u,v) \in \{(0,0),(0,1),(1,0),(1,1)\}$. It turns out that the lowest point is $(0,0)$ and the highest is $(1,0)$, that is, in the framework of (**H2**),

$$c_1^- = l_t(0,0) = 2^{-1/t}$$
$$c_1^+ = l_t(1,0) = 2.$$

5. Extensions

We are able to precisely characterize the tail of X_1+X_2 when the marginals follow a generalized logistic law and the dependence structure satisfies some technical condition. In fact, Theorem 3.2 can easily be adapted to exponential distributions. A natural question is whether other distributions with exponential tails satisfy

$$P[X_1+X_2 \leq z] \sim -czP[X_1 \leq z], \quad z \downarrow -\infty, \tag{12}$$

where $X_1 \stackrel{d}{=} X_2$, and under which dependence structure.

A good candidate is the Variance-Gamma random variable $X \stackrel{d}{=} \mathcal{VG}(\lambda,\alpha,\beta)$, with density

$$f_{VG}(x) = \frac{(\alpha^2-\beta^2)^{\lambda}|x|^{\lambda-1/2}K_{\lambda-1/2}(\alpha|x|)}{\sqrt{\pi}\Gamma(\lambda)(2\alpha)^{\lambda-1/2}}e^{\beta x}, \quad x \in \mathbb{R},$$

and moment generating function

$$M(z) = \mathbb{E}[e^{zX}] = \left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+z)^2}\right)^{\lambda}, \quad |z+\beta| < \alpha,$$

where K_ν is the modified Bessel function of the second kind of order ν. The parameter range is $\lambda > 0$, $\alpha > \beta \geq 0$. Given M, it is obvious that the Variance-Gamma class is closed under convolution and that if $X_1 \stackrel{d}{=} \mathcal{VG}(\lambda_1,\alpha,\beta)$ and $X_2 \stackrel{d}{=} \mathcal{VG}(\lambda_2,\alpha,\beta)$, then $X_1+X_2 \stackrel{d}{=} \mathcal{VG}(\lambda_1+\lambda_2,\alpha,\beta)$. We recall that

$$\int_{-\infty}^{x}(-y)^n e^{\alpha y}dy = \Gamma(n+1,\alpha x)/\alpha^{n+1}, \quad \alpha > 0,$$

where $\Gamma(\cdot,\cdot)$ is the upper incomplete gamma function. This identity, combined with he asymptotic behavior of $K_\nu(\cdot)$ and $\Gamma(\cdot,\cdot)$ given in 9.7.2 and 6.5.32 in [1] ensure that for $\lambda = 1$, (12) holds when X_1 and X_2 are independent. Whether it is possible to obtain results in the spirit of Theorem 3.2 for the $\mathcal{VG}(1,\cdot,\cdot)$ class remains an open question.

Lastly, another generalization of our result is the following. It is tempting to transform the sums in (**H1**) and (**H2**) into series. For instance, Frank's copula is given by

$$C_t^F(u,v) = -\frac{1}{t}\log\left(1+\frac{(e^{-tu}-1)(e^{-tv}-1)}{e^{-t}-1}\right), \quad t \in \mathbb{R}\backslash\{0\}$$

and

$$\frac{\partial}{\partial u}C_t^F(u,v) = \frac{e^t(e^{tv}-1)}{(e^{t(1+v)}+e^{t(1+u)}-e^t-e^{t(u+v)})^2}$$

where the denominator is bounded on $(u,v) \in [0,1]^2$ (from above by $(e^t-1)^{-1}$ and from below by $(e^{2t}-e^t)^{-1}$ for $t > 0$) and the numerator can be written

$$e^t(e^{tv}-1) = e^t\sum_{n=1}^{\infty}\frac{(tv)^n}{n!}.$$

When $t > 0$, this copula satisfies a generalized (**H2**) for $n \to \infty$. As long as $\sum_{i=1}^{\infty} c_i^+ < \infty$, it is possible to invert the integral and the series in the bounds of G and the result remains valid.

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